II.

§ 1.

$a, b, c, d \,\varepsilon\, \mathrm{q} \,.\, \supset :$

1. $a + b \,\varepsilon\, \mathrm{q} \,.$
2. $a = b \,.=.\, a + c = b + c \,.$
3. $a + b = b + a \,.$ . . . . [v. P53]
4. $(a + b) + c = a + (b + c) = a + b + c \,.$ . . . [v. P54]
5. $a + 0 = a \,.$
6. $- a \,\varepsilon\, \mathrm{q} \,.$
7. $a = b \,.=.\, - a = - b \,.$
8. $- (- a) = a \,.$
9. $a - a = 0 \,.$
10. $a = b \,.=.\, a - b = 0 \,.$
11. $- (a + b) = - a - b \,.$
12. $a - b = a + (- b) = - (b - a) \,.$
13. $a + b = c \,.=.\, a = c - b \,.$
14. $a + b = 0 \,.\, a - b = 0 \,.=.\, a = 0 \,.\, b = 0 \,.$
15. $(a - b) + b = a \,.$
16. $(a + b) - b = a \,.$
17. $a - (a - b) = b \,.$
18. $a + (b - c) = a + b - c \,.$
19. $a - (b - c) = a - b + c \,.$
20. $a - (b + c) = a - b - c \,.$
21. $a \,\varepsilon\, \mathrm{Q} \,.=.\, a > 0 \,.$
22. $a > b \,.=.\, b < a \,.=.\, a \,\varepsilon\, b + \mathrm{Q} \,.=.\, a - b \,\varepsilon\, \mathrm{Q} \,.$

23. $a > b \,.\, b > c \,.\, \supset .\, a > c$.
24. $a > b \,.\, = .\, a + c > b + c$.
25. $a > b \,.\, c > d \,.\, \supset .\, a + c > b + d$.
26. $a > b \,.\, = .\, -a < -b$.
27. $a > b \,.\, = .\, c - a < c - b$.
28. $a + a = 2a$.
29. $(a + b) + (a - b) = 2a$.
30. $(a + b) - (a - b) = 2b$.
31. $a + \mathrm{q} = \mathrm{q}$.
32. $\mathrm{q} + \mathrm{q} = \mathrm{q}$.
33. $-\mathrm{q} = \mathrm{q}$.
34. $\mathrm{Q} + \mathrm{Q} = \mathrm{Q}$.
35. $-\mathrm{Q} - \mathrm{Q} = -\mathrm{Q}$.
36. $0 \mathrel{-\varepsilon} \mathrm{Q}$.
37. $\mathrm{q} = \mathrm{Q} \cup \iota 0 \cup -\mathrm{Q}$.

41. $\mathrm{mod}\, 0 = 0$.
42. $a \,\varepsilon\, \mathrm{Q} \,.\, \supset .\, \mathrm{mod}\, a = a$. } . . . . . . [Def.]
43. $a \,\varepsilon\, -\mathrm{Q} \,.\, \supset .\, \mathrm{mod}\, a = -a$.
44. $\mathrm{mod}\,(a + b) \leqq \mathrm{mod}\, a + \mathrm{mod}\, b$ . . . . [v. P55]
45. $\mathrm{mod}\,(-a) = \mathrm{mod}\, a$.

51. $m \,\varepsilon\, \mathrm{N} \,.\, \supset .\, \mathrm{Z}_m = 1 \cup 2 \cup 3 \cup \ldots \cup m = \mathrm{N} - (m + \mathrm{N})$ . . [Def.]
52. » $.\, f \,\varepsilon\, \mathrm{q} | \mathrm{Z}_m \,.\, \supset .\, \Sigma_1^m f = \sum_{r=1}^{r=m} fr = f1 + f2 + \ldots + fm$. [Def.]
53. » . » $.\, g \,\varepsilon\, (\mathrm{Z}_m | \mathrm{Z}_m)\, \mathrm{sim} \,.\, \supset .\, \sum_{r=1}^{r=n} fr = \sum_{r=1}^{r=m} f(gr)$
54. $m, m' \varepsilon\, \mathrm{N} \,.\, f \,\varepsilon\, \mathrm{q} | \mathrm{Z}_{m+m'} \,.\, \supset .\, \sum_{r=1}^{r=n+m'} fr = \sum_{r=1}^{r=m} fr + \sum_{r=1}^{r=m'} f(m + r)$
55. $m \,\varepsilon\, \mathrm{N} \,.\, f \,\varepsilon\, \mathrm{q} | \mathrm{Z}_m \,.\, \supset .\, \mathrm{mod} \sum_{r=1}^{r=m} fr \leqq \sum_{r=1}^{r=m} \mathrm{mod}\, fr$.
56. $p, q \,\varepsilon\, \mathrm{n} \,.\, p < q . \supset . \mathrm{Z}(p, q) = p - 1 + \mathrm{Z}_{q+p} = \mathrm{n} - (q + \mathrm{N}) - (p - \mathrm{N})$ [Def.]
57. $m \,\varepsilon\, \mathrm{N} \,.\, \supset .\, \mathrm{Z}_m = \mathrm{Z}(1, m)$
58. $p, q \,\varepsilon\, \mathrm{n} \,.\, f \,\varepsilon\, \mathrm{q} | \mathrm{Z}(p, q) \,.\, \supset .\, \Sigma_p^q f = \sum_{r=p}^{r=q} fr = fp + f(p + 1) + \ldots +$
$f(q - 1) + fq$ . . . . [Def.]

§ 2.

$a, b, c, d \,\varepsilon\, q \,.\, \text{Ɔ} :$

1. $ab \,\varepsilon\, q$.
2. $a \times b = ab$.
3. $a \times 0 = 0$.
4. $a \times 1 = a$.
5. $a \times (-1) = -a$.
6. $ab = ba$ . . . . [v. P44]
7. $(ab)c = a(bc) = abc$ . . . . [v. P45]
8. $a(b + c) = ab + ac$ . . . . [v. P46]
9. $a(b - c) = ab - ac$.
10. $ab = 0 \,.\, = \,.\, a = 0 \,.\cup.\, b = 0$ . . . . [v. P47]
11. $ab -= 0 \,.\, = \,.\, a -= 0 \,.\, b -= 0$.
12. $a = b \,.\, \text{Ɔ} \,.\, ac = bc$.
13. $ac = bc \,.\, c -= 0 \,.\, \text{Ɔ} \,.\, a = b$.
14. $a, b \,\varepsilon\, Q \,.\, \text{Ɔ} \,.\, ab \,\varepsilon\, Q$.
15. $a > b \,.\, c \,\varepsilon\, Q \,.\, \text{Ɔ} \,.\, ac > bc$.
16. $c \,\varepsilon\, Q \,.\, ac > bc \,.\, \text{Ɔ} \,.\, a > b$.
17. $\mathrm{mod}\,(ab) = (\mathrm{mod}\, a)\,(\mathrm{mod}\, b)$ . . . . [v. P48]
18. $Q \times Q = Q$.
19. $Q \times (-Q) = (-Q) \times Q = -Q$.
20. $(-Q) \times (-Q) = Q$.
21. $a \,\varepsilon\, q \,.\, a -= 0 \,.\, \text{Ɔ} \,.\, /a \,\varepsilon\, q$.
22. » » $. \text{Ɔ} \,.\, /(/a) = a$.
23. » » $. \text{Ɔ} \,.\, a/a = 1$.
24. $a, b -= 0 \,.\, \text{Ɔ} \,.\, /(ab) = (/a)(/b)$.
25. $a -= 0 \,.\, \text{Ɔ} : ab = c \,.\, = \,.\, b = c/a$.
26. ~~a, b, c ε~~ b -= 0 q $. \text{Ɔ} \,.\, ab/b = a$.
27. a, b -= 0 $. \text{Ɔ} \,.\, a/(a/b) = b$.
28. b . c -= 0 $. \text{Ɔ} \,.\, a/(b/c) = ac/b$.
29. » » $. \text{Ɔ} \,.\, a/b = ac/bc$.
30. $c -= 0 \,.\, \text{Ɔ} \,.\, (a + b)/c = a/c + b/c$.
a -= 0 Ɔ . 31. $\mathrm{mod}\, /a = /\mathrm{mod}\, a$.

32. $a, b \varepsilon \mathrm{Q} . \supset : a > b . = . /a < /b$.
33. » $. a > b . = . a/b > 1$.
34. $/-\mathrm{Q} = -/\mathrm{Q} = -\mathrm{Q}$.
35. $b -= 0 . \supset . a/b = c + (a - bc)/b = c - (bc - a)/b$.
36. $a/b = c/d . = . a/c = b/d$.
37. » $. = . (a + b)/b = (c + d)/d$.
38. $a > b . \supset : a/b = c/d . = . (a - b)/b = (c - d)/d . = . (a + b)/(a - b) = (c + d)/(c - d)$.
39. $a/b = c/d = e/f . \supset . a/b = (a + c + e)/(b + d + f)$.
40. $a/b = d/e . b/c = e/f . \supset . a/c = d/f$.
41. $a/b = e/f . b/c = d/e . \supset . a/c = d/f$.
42. $a/b = c/d . e/b = f/d . \supset . (a + e)/b = (c + f)/d$. 42' a/b = c/d . = . ad = bc
43. $m \varepsilon \mathrm{N} . f \varepsilon \mathrm{q}/\mathrm{Z}_m . \supset . \Pi_1^m f = \Pi_{r=1}^{r=m} fr = (f1) \times (f2) \times \ldots \times (fm)$ [Def.]
44. » » $. g \varepsilon (\mathrm{Z}_n/\mathrm{Z}_m) \operatorname{sim} . \supset . \Pi_{r=1}^{r=m} fr = \Pi_{r=1}^{r=m} f(gr)$
45. $m, m' \varepsilon \mathrm{N} . f \varepsilon \mathrm{q}/\mathrm{Z}_{m+m'} . \supset . \Pi_{r=1}^{r=m+m'} fr = \Pi_{r=1}^{r=m} fr \times \Pi_{r=1}^{r=m'} f(m + r)$
46. $m \varepsilon \mathrm{N} . f \varepsilon \mathrm{q}/\mathrm{Z}_m . \supset . a \Sigma_{r=1}^{r=m} fr = \Sigma_{r=1}^{r=m} afr$.
47. » » $. \Pi_1^m f = 0 . \supset . 0 \varepsilon f\mathrm{Z}_m$.
48. » » $. \supset . \operatorname{mod} \Pi_{r=1}^{r=m} fr = \Pi_{r=1}^{r=m} \operatorname{mod} fr$.

§ 3.

$a, b, c, d, a', \ldots d' \varepsilon \mathrm{q} . \supset :$

1. $m \varepsilon \mathrm{N} . \supset . a^m \varepsilon \mathrm{q}$.
2. » $1^m = 1$.
3. » $0^m = 0$.
4. $a^1 = a$.
5. $a -= 0 . m \varepsilon - \mathrm{N} . \supset . a^m = /a^{-m}$ . . . . [Def.]
6. » $. \supset . a^0 = 1$ . . . . [Def.]

$m, n \varepsilon \mathrm{N} . \supset :$

7. $a^m a^n = a^{m+n}$.
8. $(a^m)^n = a^{mn}$.
9. $(ab)^m = a^m b^m$ . . . . [v. P18]

10. $a -= 0 \,.\, b -= 0 \,.\, m, n \,\varepsilon\, \mathrm{n} \,.\, ɔ \,.\, \mathrm{P7} \,.\, \mathrm{P8} \,.\, \mathrm{P9}$.
11. $a -= 0 \,.\, m \,\varepsilon\, \mathrm{n} \,.\, ɔ \,.\, (/a)^m = /a^m = a^{-m}$.
12. » $.\, ɔ \,.\, \mathrm{mod}\,(a^m) = (\mathrm{mod}\, a)^m$.
13. » $.\, ɔ \,.\, a^{2m} = (\mathrm{mod}\, a)^{2m}$.
14. $a \,\varepsilon\, -\mathrm{Q} \,.\, ɔ \,.\, a^{2m+1} = -(\mathrm{mod}\, a)^{2m+1}$.
15. $a, b \,\varepsilon\, \mathrm{Q} \,.\, m \,\varepsilon\, \mathrm{N} \,.\, ɔ : a \leqq b \,.=.\, a^m \leqq b^m$.
16. $a \,\varepsilon\, \mathrm{Q} \,.\, a > 1 \,.\, m, n \,\varepsilon\, \mathrm{N} \,.\, ɔ : m < n \,.=.\, a^m < a^n$.
17. » $.\, a < 1 \,.$ » $.\, ɔ : m < n \,.=.\, a^m > a^n$.
18. $m \,\varepsilon\, \mathrm{N} \,.\, f \,\varepsilon\, \mathrm{q}/\mathrm{Z}_m \,.\, ɔ \,.\, (\Pi_{r=1}^{r=m} fr)^m = \Pi_{r=1}^{r=m} (fr)^m$

§ 4.

1. $(a + b)(c + d) = ac + ad + bc + bd$.
2. $(a - b)(c - d) = (ac + bd) - (ad + bc)$.
3. $(a + b)(a - b) = a^2 - b^2$.
4. $(a + b)^2 = a^2 + 2ab + b^2$.
5. $(a - b)^2 = a^2 - 2ab + b^2$.
6. $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$
7. $(a + b)^2 - (a - b)^2 = 4ab$.
8. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.
9. $a(b - c) + b(c - a) + c(a - b) = 0$.
10. $(a - b)(c - d) + (b - c)(a - d) + (c - a)(b - d) = 0$.
11. $(a+b+c+d)^2 + (a+b-c-d)^2 + (a+c-b-d)^2 + (a+d-b-c)^2 =$
$(-a+b+c+d)^2 + (a-b+c+d)^2 + (a+b-c+d)^2 + (a+b+c-d)^2 =$
$4(a^2+b^2+c^2+d^2)$.
12. $(a-b)^2+(b-c)^2+(c-a)^2 = 2[(a-b)(a-c)+(b-a)(b-c)+(c-a)(c-b)]$.
13. » » » $= 2(a^2+b^2+c^2-ab-ac-bc)$
14. $(a+b)^2 + a^2 + b^2 = 2(a^2 + ab + b^2)$
21. $(a^2 + ab + b^2)(a - b) = a^3 - b^3$.
22. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
23. » $= a^3 + b^3 + 3ab(a + b)$.
24. $(a+b+c)^3 = a^3+b^3+c^3+3(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2)+6abc$.
25. » $= a^3 + b^3 + c^3 + 3(a + b)(a + c)(b + c)$.
26. $a^2(b - c) + b^2(c - a) + c^2(a - b) = (a - b)(a - c)(b - c)$.
15. $4(a^2+ab+b^2) = 3(a+b)^2 + (a-b)^2$

27. $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2) = (b - a)(c - a)(c - b)$.

28. $(a + b + c)^3 - (b + c - a)^3 - (c + a - b)^3 - (a + b - c)^3 = 24abc$.

29. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc)$

30. » $= (a + b + c)[(a - b)^2 + (b - c)^2 + (c - a)^2]/2$

31. $(a + b + c)^3 - a^3 - b^3 - c^3 = 3(a + b)(a + c)(b + c)$.

32. » $= 3(a + b + c)(ab + ac + bc) - 3abc$

33. $(a + b - c)(a - b + c)(-a + b + c) = a^2(b + c - a) + b^2(a + c - b)$
$+ c^2(a + b - c) - 2abc$.

34. $a + b + c = 0 \,.\supset.\, a^3 + b^3 + c^3 = 3abc$.

35. $a^3 + b^3 + c^3 = 3abc \,.\supset.\, a = b = c \,.\cup.\, a + b + c = 0$.

36. $(a - b)^3 + (b - c)^3 + (c - a)^3 = 3(a - b)(b - c)(c - a)$.

37. $4(a^3 + b^3) = (a + b)^3 + 3(a + b)(a - b)^2$

40. $(a^3 + a^2b + ab^2 + b^3)(a - b) = a^4 - b^4$.

41. $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

42. $(a + b + c)^4 = a^4 + b^4 + c^4 + 4(a^3b + a^3c + b^3a + b^3c + c^3a + c^3b)$
$+ 6(a^2b^2 + a^2c^2 + b^2c^2) + 12(a^2bc + b^2ac + c^2ab)$

43. » $= 2(a^2 + b^2 + c^2)(a + b + c)^2 + 8abc(a + b + c)$
$+ 2(a^2b^2 + a^2c^2 + b^2c^2) - (a^4 + b^4 + c^4)$

44. $a(b-c)^3 + b(c-a)^3 + c(a-b)^3 = (a-b)(b-c)(c-a)(a+b+c)$.

45. $a^3(b-c) + b^3(c-a) + c^3(a-b) = (a-b)(a-c)(b-c)(a+b+c)$.

46. $(a + b + c)(a + b - c)(a - b + c)(-a + b + c) = 2(a^2b^2 + a^2c^2 + b^2c^2) -$
$(a^4 + b^4 + c^4)$.

47. $(a^2 + b^2)(a'^2 + b'^2) = (aa' + bb')^2 + (ab' - a'b)^2 = (aa' - bb')^2 + (ab' + a'b)^2$.

48. $(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2$

49. $(a^2 + cb^2)(a'^2 + cb'^2) = (aa' + cbb')^2 + c(ab' - a'b)^2$.

50. » » $= (aa' - cbb')^2 + c(ab' + a'b)^2$

51. $(a^2 + b^2 + c^2)(a'^2 + b'^2 + c'^2) - (aa' + bb' + cc')^2 = (ab' - a'b)^2 + (ac' - a'c)^2 +$
$(bc' - b'c)^2$.

52. $(a^2 + b^2 + c^2 + d^2)(a'^2 + b'^2 + c'^2 + d'^2) = (aa' + bb' + cc' + dd')^2 + (ab' - a'b +$
$cd' - c'd)^2 + (ac' - a'c + bd' - b'd)^2 + (ad' - a'd + bc' - b'c)^2$.

53. $a^2(a + b)^2 + a^2b^2 + (a + b)^2b^2 = (a^2 + ab + b^2)^2$.

54. $[(a - b)^2 + (b - c)^2 + (c - a)^2]^2 = 2[(a - b)^4 + (b - c)^4 + (c - a)^4]$

55. $(a+b)^4 + (a+c)^4 + (b+c)^4 = (a+b+c)^4 + a^4 + b^4 + c^4 + 12abc(a+b+c)$

56. $(a^2 - b^2)(c^2 - d^2) = (ac - bd)^2 - (ad - bc)^2$.

57. $(a^2 + ab + b^2)^2 - (a^2 - ab + b^2)^2 = 4ab(a^2 + b^2)$.

38. $a^3 + a^2b + ab^2 + b^3 = (a + b)(a^2 + b^2)$

39. " " $= \frac{1}{2}(a + b)^3 + \frac{1}{2}(a + b)(a - b)^2$

52'. $(a^2 - pb^2 - qc^2 + pq\,d^2)(a'^2 - pb'^2 - qc'^2 + pq\,d'^2) = (aa' + pbb' \pm q(cc' + dd'))^2$
$- p(ab' + a'b \pm q(cd' + c'd))^2 - q(ac' - pbd' \pm (a'c - pb'd))^2$
$+ pq(bc' - ad' \pm (a'd - b'c))^2$ [Lagr. Nouv. Mém. de B. 1770 p. 133
Beccari, ... p. 2017.

58. $a(a-2b)^3 - b(b-2a)^3 = (a-b)(a+b)^3$.

59. $a^4 + 4b^4 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$.

60. $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$.

61. $a^3(b^2-c^2) + b^3(c^2-a^2) + c^3(a^2-b^2) = (a-b)(a-c)(b-c)(ab+ac+bc)$.

62. $(a+b)^5 - a^5 - b^5 = 5ab\,(a+b)\,(a^2+ab+b^2)$.

63. $(a+b)^7 - a^7 - b^7 = 7ab(a+b)\,(a^2+ab+b^2)^2$.

64. $a, b, c, d \,\varepsilon\, \mathrm{Q} . \supset : a/b = c/d .=. (a+b)^2/(c+d)^2 = (a^2+b^2)/(c^2+d^2)$.

65. $(a-c)/(c-b) = a/b .=. c = 2ab/(a+b) .=. /c = \frac{1}{2}(/a + /b)$.

§ 5.

1. $a \,\varepsilon\, \mathrm{q} . a -= 0 . \supset . a^2 > 0$.

2. $a, b, a', b' \,\varepsilon\, \mathrm{q} . a > b . a' > b' . \supset . aa' + bb' > ab' + a'b$.

[Hp. $\supset . (a-b)(a'-b') > 0 . \supset .$ Ts.]

3. $a, b \,\varepsilon\, \mathrm{Q} . a -= b . \supset . a^2 + b^2 > 2ab$. [Hp. $\supset . (a-b)^2 > 0 . \supset .$ Ts.]

4. $(a^2 + ab + b^2)^2 < 3(a^4 + a^2b^2 + b^4)$.

[$3(a^4 + a^2b^2 + b^2) - (a^2 + ab + b^2)^2 = 2(a-b)^2(a^2 + ab + b^2)$]

$a, b, c \,\varepsilon\, \mathrm{Q} . -(a = b = c) . \supset :$

11. $a^2 + b^2 + c^2 > ab + ac + bc$.

[Hp. $\supset . (a-b)^2 + (a-c)^2 + (b-c)^2 > 0 . \supset .$ Ts.]

12. $(a+b+c)^2 < 3(a^2 + b^2 + c^2)$.

13. $(a+b-c)^2 + (a+c-b)^2 + (b+c-a)^2 > ab + bc + ca$.

14. $abc > (a+b-c)(a+c-b)(b+c-a)$.

15. $2(a^3 + b^3 + c^3) > ab(a+b) + ac(a+c) + bc(b+c) > 6abc$.

16. $(a+b)(b+c)(c+a) > 8abc$.

17. $3(a^3 + b^3 + c^3) > (a+b+c)(ab+bc+ca)$.

18. $9abc < (a+b+c)(a^2+b^2+c^2)$.

19. $8(a^3 + b^3 + c^3) > 3(a+b)(b+c)(c+a)$.

20. $27abc < (a+b+c)^3 < 9(a^3+b^3+c^3)$.

21. $a, b, c \,\varepsilon\, \mathrm{Q} . a < b < c . a + b > c . \supset . 2(ab + ac + bc) > a^2 + b^2 + c^2$.

22. $a>b>c . \cup . b>c>a . \cup . c>a>b . \supset . a^2b + b^2c + c^2a < a^2c + b^2a + c^2b$.

23. $a^4 + b^4 + c^4 > abc(a+b+c)$.

24. $a/b = c/d \,.\, a > b \,.\, a > c \,.\, \supset .\, a + d > b + c$.

25. $a/b < c/d \,.\, \supset .\, \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

26. $a/(a+b) < (a+c)/(a+b+c)$

27. $a -= b \,.\, \supset .\, \left(\frac{2ab}{a+b}\right)^2 < ab < \left(\frac{a+b}{2}\right)$.

§ 6.

$a, b \,\varepsilon\, \mathrm{Q} \,.\, m, n, m', n' \,\varepsilon\, \mathrm{N} \,.\, \supset :$

1. $\sqrt[m]{a} \,\varepsilon\, \mathrm{Q}$.
2. $(\sqrt[m]{a})^m = \sqrt[m]{a^m} = a$.
3. $\sqrt{a} = \sqrt[2]{a}$. $\quad \sqrt[1]{a} = a$.
4. $\sqrt[m]{ab} = \sqrt[m]{a}\sqrt[m]{b}$.
5. $\sqrt[m]{/a} = /\sqrt[m]{a}$.
6. $(\sqrt[m]{a})^n = \sqrt[m]{a^n}$.
7. $\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$.
8. $\frac{m}{n} = \frac{m'}{n'} \,.\, \supset .\, \sqrt[m]{a^n} = \sqrt[m']{a^{n'}}$.

11. $\sqrt{a} - \sqrt{b} < \sqrt{a+b} < \sqrt{a} + \sqrt{b}$.

12. $\sqrt{a+b} > \sqrt{a} + \sqrt{b} - \sqrt[4]{\frac{ab}{4}}$

$$\left[\sqrt{a} + \sqrt{b} - \sqrt{a+b} = \frac{2\sqrt{ab}}{\sqrt{a} + \sqrt{b} + \sqrt{a+b}} < \frac{2\sqrt{ab}}{2\sqrt{a+b}} < \sqrt{\frac{ab}{2\sqrt{ab}}}\right]$$

13. $a -= b \,.\, \supset .\, \frac{a+b}{2} > \sqrt{ab}$. [Hp. $\supset .\, (\sqrt{a} - \sqrt{b})^2 > 0 \,.\, \supset .$ Ts.]

14. $a > b \,.\, \supset .\, \sqrt{ab} + \frac{(a-b)^2}{8b} > \frac{a+b}{2} > \sqrt{ab} + \frac{(a-b)^2}{8a}$

21. $a, b, a', b' \varepsilon \mathrm{R} \,.\, b - \varepsilon \mathrm{R}^2 \,.\, a + \sqrt{b} = a' + \sqrt{b'} \,.\supset .\, a = a' \,.\, b = b'$.

22. $a > \sqrt{b} \,.\supset .\, \sqrt{a + \sqrt{b}} = \sqrt{\dfrac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\dfrac{a - \sqrt{a^2 - b}}{2}}$.

23. » $.\supset .\, \sqrt{a - \sqrt{b}} =$ » $-$ »

24. $\left[\sqrt{a^2 + \sqrt[3]{a^4 b^2}} + \sqrt{b^2 + \sqrt[3]{a^2 b^4}}\right]^2 = [\sqrt[3]{a^2} + \sqrt[3]{b^2}]^3$.

25. $b \leqq 8a \,.\supset .\, \sqrt[3]{a + (b+a)\sqrt{\dfrac{8a-b}{27b}}} + \sqrt[3]{a - (b+a)\sqrt{\dfrac{8a-b}{27b}}} = \sqrt[3]{b}$.

26. $a, b \,\varepsilon\, \mathrm{Q} \,.\, b \leqq a \,.\supset .\, \sqrt[4]{\dfrac{2a - b + 2\sqrt{a(a-b)}}{4}} + \sqrt[4]{\dfrac{2a - b - 2\sqrt{a(a-b)}}{4}}$

$= \sqrt{\sqrt{a} + \sqrt{b}}$.

§ 7.

1. $a \,\varepsilon\, \mathrm{Q} \,.\, x \,\varepsilon\, \mathrm{q} \,.\supset .\, a^x \,\varepsilon\, \mathrm{Q}$.
2. » $.\, x, y \,\varepsilon\, \mathrm{q} \,.\supset .\, a^{x+y} = a^x a^y$.
3. » » $.\supset .\, (a^x)^y = a^{xy}$.
4. » $.\supset .\, a^0 = 1$.
5. » $.\supset .\, a^{-x} = \dfrac{1}{a^x}$.
6. » $m, n \,\varepsilon\, \mathrm{N} \,.\supset .\, a^{\frac{m}{n}} = \sqrt[n]{a^m}$.
7. $a > 1 \,.\supset :\, x < y \,.=.\, a^x < a^y$.
8. $a \,\varepsilon\, \mathrm{Q} \,.\, m \,\varepsilon\, \mathrm{Q} \,.\, m > 1 \,.\supset .\, (1+a)^m > 1 + ma$.
9. » » $.\, m < 1 \,.\supset .\, (1+a)^m < 1 + ma$. $\left[\begin{pmatrix} 1/m, & ma \\ m, & a \end{pmatrix} \mathrm{P}8 \supset \mathrm{P}9\right]$
10. $a \,\varepsilon\, \mathrm{Q} . m, n \,\varepsilon\, \mathrm{Q} . m > n .\supset . \left(1 + \dfrac{a}{m}\right)^m > \left(1 + \dfrac{a}{n}\right)^n$. $\left[\begin{pmatrix} m/n, & a/m \\ m, & a \end{pmatrix} \mathrm{P}8 \supset \mathrm{P}10\right]$
11. $m, n \,\varepsilon\, \mathrm{Q} \,.\, m > n \,.\supset .\, \left(1 + \dfrac{1}{m}\right)^m > \left(1 + \dfrac{1}{n}\right)^n$. $\left[\begin{pmatrix} 1 \\ a \end{pmatrix} \mathrm{P}10 \supset \mathrm{P}11\right]$
12. $a \,\varepsilon\, \mathrm{Q} \,.\, m \,\varepsilon\, \mathrm{Q} \,.\supset .\, (1+a)^m > 1 + \dfrac{ma}{1+a}$.

   [Hp. P8 $.\supset .\, (1+a)^{m+1} > 1 + a + ma \,.\supset .$ Ts.]
13. $m, n \,\varepsilon\, \mathrm{Q} \,.\supset .\, \left(\dfrac{m+1}{m}\right)^m < \left(\dfrac{n+1}{n}\right)^{n+1}$. $\left[\begin{pmatrix} 1/n, & (n+1)/m \\ a, & m \end{pmatrix} \mathrm{P}12 \supset \mathrm{P}13\right]$

21. $a \varepsilon \mathrm{Q} . a -= 1 . x \varepsilon \mathrm{Q} . y \varepsilon \mathrm{q} . \supset : y = \mathrm{Log}_a x . = a^y = x .$
22. $a \varepsilon \mathrm{Q} . a -= 1 . x \varepsilon \mathrm{Q} . \supset . \mathrm{Log}_a x \varepsilon \mathrm{q} .$
23. » » $. \supset . a^{\mathrm{Log}_a x} = x .$
24. » $. x \varepsilon \mathrm{q} . \supset . \mathrm{Log}_a a^x = x .$
25. $a, b \varepsilon \mathrm{Q} . a -= 1 . b -= 1 . x \varepsilon \mathrm{Q} . \supset . \mathrm{Log}_a x = \mathrm{Log}_b x \times \mathrm{Log}_a b .$
26. » » » $. \supset . 1 = \mathrm{Log}_a b \, \mathrm{Log}_b a$

$a \varepsilon \mathrm{Q} . a -= 1 . \mathrm{Log}\, x = \mathrm{Log}_a x . \supset :$

27. $\mathrm{Log}\, 1 = 0$
28. $\mathrm{Log}\, a = 1 .$
29. $x, y \varepsilon \mathrm{Q} . \supset . \mathrm{Log}\, (xy) = \mathrm{Log}\, x + \mathrm{Log}\, y .$
30. » $\mathrm{Log}\, \frac{x}{y} = \mathrm{Log}\, x - \mathrm{Log}\, y .$
31. $x \varepsilon \mathrm{Q} . m \varepsilon \mathrm{q} . \supset . \mathrm{Log}\, x^m = m \, \mathrm{Log}\, x .$

§ 8.

$a, b, c, d, a', b', p, q, x, y, z \varepsilon \mathrm{q} . \supset :$

1. $x + a = b . = . x = b - a .$
2. $a -= 0 . \supset : ax = b . = . x = b/a .$
3. $a = 0 . b -= 0 . \supset : ax = b . =_x . \Lambda$
4. $a = 0 . b = 0 . \supset . ax = b .$
5. $ax + b = a'x + b' . = . (a - a')x = b' - b .$
6. $x + y = a . x - y = b . = . x = (a + b)/2 . y = (a - b)/2 .$
7. $p, q, p+q -= 0 . \supset : x + y = a . x/p = y/q . = . x = pa/(p+q) . y = qa/(p+q) .$
8. $y + z = a . z + x = b . x + y = c . = . x = (b + c - a)/2 . y = (a + c - b)/2 . z = (a + b - c)/2 .$
9. $y + z - x = a . z + x - y = b . x + y - z = c . = . x = (b + c)/2 . y = \ldots$
10. $ab' - a'b -= 0 . \supset : ax + by = c . a'x + b'y = c' . = . x = (cb' - c'b)/(ab' - a'b) . y = (ac' - a'c)/(ab' - a'b) .$
11. $x, y \varepsilon \mathrm{q} . -(x=0 . y=0) . ax + by = 0 . a'x + b'y = 0 . -=_{x,\, y} \Lambda : = : ab' - a'b = 0 .$
12. $(a-b)(a-c)(b-c) -= 0 . \supset : x + y + z = 1 . ax + by + cz = d . a^2x + b^2y + c^2z = d^2 . = . x = \frac{(d-b)(d-c)}{(a-b)(a-c)} . y = \ldots$
13. $abc -= 0 . \supset : xy = a^2 . yz = b^2 . zx = c^2 . = . x = \frac{ac}{\pm b} . y = \frac{ab}{\pm c} . z = \frac{bc}{\pm a} .$

20. $\begin{cases} x^2 = a^2 . = . x = a . \cup . x = -a . \\ \quad » \quad = . x = \pm a . \end{cases}$

21. $\begin{cases} a > 0 . \supset : x^2 = a . = . x = \pm \sqrt{a} . \\ a = 0 . \supset : x^2 = a . = . x = 0 . \\ a < 0 . \supset : x^2 = a . =_x \Lambda . \end{cases}$

22. $x^2 + px + q = \left(x + \frac{p}{2}\right)^2 + q - \frac{p^2}{4} .$

23. $\begin{cases} \frac{p^2}{4} - q > 0 . \supset : x^2 + px + q = 0 . = . x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q} . \\ \quad » \quad = 0 . \supset : \quad » \quad = . x = -\frac{p}{2} . \\ \quad » \quad < 0 . \supset : \quad » \quad =_x \Lambda . \end{cases}$

24. $\begin{cases} a -= 0 . b^2 - ac > 0 . \supset : ax^2 + 2bx + c = 0 . = . x = \frac{-b \pm \sqrt{b^2 - ac}}{a} . \\ \quad » \quad » \quad = 0 . \supset : \quad » \quad = . x = -\frac{b}{a} . \\ \quad » \quad » \quad < 0 . \supset : \quad » \quad =_x \Lambda . \end{cases}$

25. $x + y = a . xy = b . = . x + y = a . (x - y)^2 = a^2 - 4b .$

26. $x + y = a . x^2 + y^2 = b^2 . = . x + y = a . 2xy = a^2 - b^2 .$

27. $x + y = a . x^2 - y^2 = b . = . x + y = a . a(x - y) = b .$

28. $x^2 + y^2 = a . xy = b . = . (x + y)^2 = a + 2b . (x - y)^2 = a - 2b .$

29. $x^3 + y^3 = a . x + y = b . = . x + y = b . xy = \frac{b^3 - a}{3b} .$

30. $x^4 + y^4 = a . x + y = b . = . x + y = b . 2x^2y^2 - 4b^2xy + b^4 - a = 0 .$

31. $x^5 + y^5 = a . x + y = b . = . x + y = b . 5bxy(b^2 - xy) = b^5 - a .$

§ 9.

1. $\mathrm{i} \,\varepsilon\, \mathrm{q}' .$
2. $\mathrm{i}^2 = -1$
3. $\mathrm{q}' = \mathrm{q} + \mathrm{iq} .$
4. $x, y, x', y' \,\varepsilon\, \mathrm{q} . \supset : x + \mathrm{i}y = x' + \mathrm{i}y' . = . x = x' . y = y' .$
5. $\quad » \quad . \supset . (x + \mathrm{i}y) + (x' + \mathrm{i}y') = (x + x') + \mathrm{i}(y + y') .$
6. $\quad » \quad . \supset . (x + \mathrm{i}y)(x' + \mathrm{i}y') = (xx' - yy') + \mathrm{i}(xy' + x'y) .$
7. $x, y \,\varepsilon\, \mathrm{q} . \supset . \mathrm{mod}\,(x + \mathrm{i}y) = \sqrt{x^2 + y^2} .$
8. $\quad » \quad . x^2 + y^2 -= 0 . \supset . /(x + \mathrm{i}y) = (x - \mathrm{i}y)/(x^2 + y^2) .$
9. $\mathrm{q} \supset \mathrm{q}' .$

10. $\left(\begin{matrix} q' \\ q \end{matrix}\right)$ [§1P1-20, 28-30, 41, 44, 45, 53-55; §2P1-13, 17, 21-31, 35-48; §3 P1-13, 18; §4P1-34, 36-63; §8P1-13, 25-31].

11. $a \, \varepsilon \, q' . m \, \varepsilon \, N . \supset . \sqrt[m]{}^{*}a = q' \cap \overline{x \, \varepsilon} \, (x^m = a)$. [Def.]

12. $\sqrt[m]{}^{*}0 = 0$.

12'. $a \, \varepsilon \, q' . a -= 0 . m \, \varepsilon \, N . \supset . \mathrm{num} \sqrt[m]{}^{*}a = m$.

13. $a \, \varepsilon \, Q . \supset . \sqrt[m]{}^{*}a = \sqrt[m]{a} \sqrt[m]{}^{*}1$.

14. $a \, \varepsilon - Q . \supset . \sqrt[m]{}^{*}a = \sqrt[m]{(\mathrm{mod}\, a)} \sqrt[m]{}^{*}(-1)$.

15. $\sqrt{}^{*}1 = 1 , -1 . \sqrt{}^{*}(-1) = i , -i$.

16. $\sqrt[3]{}^{*}1 = 1 , \frac{-1 \pm i\sqrt{3}}{2}$.

17. $1 n \, \varepsilon \, N . \supset . \sqrt[2n+1]{}^{*}(-1) = -\sqrt[2n+1]{}^{*}1$

18. $\sqrt[4]{}^{*}1 = 1, -1, i, -i$

19. $\sqrt[4]{}^{*}(-1) = \frac{\pm\sqrt{2} \pm i\sqrt{2}}{2}$

20. $\sqrt[5]{}^{*}1 = 1, \frac{-1+\sqrt{5} \pm i\sqrt{10-2\sqrt{5}}}{4}, \frac{-1-\sqrt{5} \pm i\sqrt{10+2\sqrt{5}}}{4}$

21. $\sqrt[6]{}^{*}1 = \sqrt[3]{}^{*}1 \cup \sqrt[3]{}^{*}-1$

22. $\sqrt[6]{}^{*}(-1) = \pm i, \frac{\pm\sqrt{3} \pm i}{2}$

23. $x \varepsilon q . y \varepsilon Q . \supset . \sqrt{}^{*}(x+iy) = \pm\left(\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + i\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right)$

24. » $\sqrt{}^{*}(x-iy) = \pm$ » — »

25. $a -= 0 . \supset : ax^2 + bx + c = 0 . = . x = \frac{-b + \sqrt{}^{*}b^2 - 4ac}{2a}$.

26. » $. \supset : ax^4 + bx^2 + c = 0 . = . x = \sqrt{}^{*}\frac{-b + \sqrt{}^{*}b^2 - 4ac}{2a}$.

§ 10.

$a, b, c, x \,\varepsilon\, q \,.\, m \,\varepsilon\, N \,.\, \supset :$

1. $x -= 1 \,.\, \supset \,.\, \sum_{r=0}^{r=m} ax^r = a + ax + ax^2 + \ldots + ax^m = a \frac{x^{m+1} - 1}{x - 1}$.

2. $a -= b \,.\, \supset \,.\, \sum_{r=0}^{r=m} a^{m-r} b^r = a^m + a^{m-1} b + \ldots + b^m = \frac{a^{m+1} - b^{m+1}}{a - b}$.

3. $(a+b)^m = \Sigma_{r=0}^{r=m} \frac{m!}{r!\,(m-r)!} a^{m-r} b^r$.

$$= a^m + ma^{m-1} b + \frac{m(m-1)}{1.2} a^{m-2} b^2 + \ldots + mab^{m-1} + b^m .$$

4. $(a+b+c)^m = \Sigma_{r=0}^{r=m} \Sigma_{s=0}^{s=m-r} \frac{m!}{r!\,s!\,(m-r-s)!} a^r b^s c^{m-r-s}$.

5. $\sum_{r=1}^{r=m} r = 1 + 2 + \ldots + m = m(m+1)/2$.

6. $\Sigma r^2 = 1^2 + 2^2 + \ldots + m^2 = m(m+1)(2m+1)/6$

7. $\Sigma r^3 = m^2(m+1)^2/4 = (\Sigma r)^2$.

8. $\Sigma r^4 = m^5/5 + m^4/2 + m^3/3 - m/30$.

9. $\Sigma r^5 = m^6/6 + m^5/2 + 5m^4/12 - m/12$. etc.

10. $\Sigma_{r=1}^{r=m} r(r+1) = m(m+1)(m+2)/3$.

11. $\Sigma r(r+1)(r+2) = m(m+1)(m+2)(m+3)/4$.

12. $\Sigma_{r=1}^{r=m} \Pi_{s=0}^{s=n} (r+s) = [\Pi_{s=0}^{s=n+1} (m+s)]/(n+2)$.

13. $\Sigma_{r=0}^{r=n-1} (a+br) = na + \frac{n(n-1)}{2} b$.

14. $\Sigma_{r=0}^{r=n-1} (2r+1) = 1 + 3 + 5 + \ldots + (2n-1) = n^2$.

$n \,\varepsilon\, N \,.\, x_1, x_2, \ldots x_n, y_1, y_2, \ldots y_n, m \,\varepsilon\, Q \,.\, \supset :$

15. $x_1 + x_2 + \ldots + x_n \geq n \sqrt[n]{x_1 x_2 \ldots x_n}$

16. $\frac{n-1}{2}(x_1 + x_2 + \ldots + x_n) \geq \sqrt{x_1 x_2} + \sqrt{x_1 x_3} + \sqrt{x_2 x_3} + \ldots + \sqrt{x_{n-1} x_n}$

17. $x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \leq \sqrt{(x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2)}$

18. $1 \,.\, 2 \,.\, 3 \ldots n \leq \left(\frac{n+1}{2}\right)^n$

19. $1 \,.\, 3 \,.\, 5 \ldots (2n-1) \leq n^n$

20. $\left(\frac{x_1}{y_1}\right)^{y_1}\left(\frac{x_2}{y_2}\right)^{y_2}\dots\left(\frac{x_n}{y_n}\right)^{y_n} \leqq \left(\frac{x_1+x_2+\dots+x_n}{y_1+y_2+\dots+y_n}\right)^{y_1+y_2+\dots+y_n}$

21. $x_1^{x_1}x_2^{x_2}\dots x_n^{x_n} \leqq \left(\frac{x_1+x_2+\dots+x_n}{n}\right)^{x_1+x_2+\dots+x_n}$

22. $x_1^{x_1y_1}x_2^{x_2y_2}\dots x_n^{x_ny_n} \geqq \left(\frac{x_1y_1+x_2y_2+\dots+x_ny_n}{y_1+y_2+\dots+y_n}\right)^{x_1y_1+x_2y_2+\dots+x_ny_n}$

23. $m > 1 . \supset : \frac{y_1x_1^m+y_2x_2^m+\dots+y_nx_n^m}{y_1+y_2+\dots+y_n} \geqq \left(\frac{x_1y_1+x_2y_2+\dots+x_ny_n}{y_1+y_2\dots+y_n}\right)^m$

24. $m < 1 . \supset :$ » $\leqq$ »